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THE THEORY OF VERY SHALLOW SHELLS, BASED ON THE ASYMMETRIC THEORY OF ELASTICITY†

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A method of reducing a three-dimensional problem in the asymmetric theory of elasticity to a two-dimensional problem in the theory of shells is proposed. © 1998 Elsevier Science Ltd. All rights reserved.

The proposed approximate theory starts from the ideas of the improved theory of shells [1–3] and the general asymmetric theory of elasticity [4–6]. It can be used not only to analyse very shallow shells [3], but also to investigate shells with a large variability index, to construct a simple edge effect, to analyse local stability problems, etc.

The theory may be of interest when considering shells made of polycrystalline materials, high polymers, grainy composites, etc. The results can be used when investigating microshells and plates encountered in micromolecular mechanics.

1. Consider a shell of constant thickness h in an orthogonal curvilinear system of coordinates α_i. The middle surface of the shell is the coordinate surface $\alpha_1\alpha_2$. The coordinate line α_3 is rectilinear. The coordinate lines α_1 and α_2 coincide with the lines of curvature of the middle surface. The principal curvatures of the middle surface and the coefficient of its first quadratic form are, respectively, $k_i = k_i(\alpha_1, \alpha_2)$ and $A_i = A_i(\alpha_1, \alpha_2)$, but are assumed to behave like constants during differentiation [1–3]. The system of coordinates is chosen so that the strong inequality $AB/R_1R_2 \ll 1$ holds (R_i are the principal radii of curvature of the middle surface) [3]. The shell is assumed to be loaded only by surface forces applied at right angles with intensities Z^+ (for $\alpha_3 = h/2$) and Z^- (for $\alpha_3 = -h/2$).

The theory is based on the following hypotheses [1–3].

1. The displacement u_3, perpendicular to the middle surface of the shell and rotations about the normal lines $\alpha_3 - \omega_3$ are independent of the coordinate α_3 [1–3], that is

$$u_3 = w(\alpha_1, \alpha_2), \quad \omega_3 = \psi_3(\alpha_1, \alpha_2) \tag{1.1}$$

where w is the required normal displacement of the middle surface and ψ_3 is the required rotation about the coordinates α_3.

2. Over the thickness of the shell, the shear stresses σ_{31} and σ_{32} behave in accordance with the law described in [1, 2], that is

$$\sigma_{3n} = f(\alpha_3)\varphi_n(\alpha_1, \alpha_2) \qquad n = 1, 2 \tag{1.2}$$

where φ_n are the unknown functions and $f = (h^2/4 - \alpha_3^2)/2$ is a given function for which the conditions $\alpha_3 = \pm h/2$ are satisfied by the stresses σ_{3n}.

3. The structure of the rotations ω_1 and ω_2 is determined by the improved theory of very shallow shells [1, 2], that is

$$\omega_1 = w_{,2} - f(\alpha_3)\psi_2, \quad \omega_2 = -w_{,1} + f(\alpha_3)\psi_2 \tag{1.3}$$

where $\psi_n(\alpha_1, \alpha_2)$ are the required functions.

4. The normal stresses σ_{33} can be neglected compared with the stresses σ_{11} and σ_{22}. The stress σ_{33} can be determined from the equilibrium equation.

Without loss of generality, we will assume that $A_i = 1$ and the Lamé coefficients are given by $H_1 = 1 + k_1\alpha_3$, $H_2 = 1 + k_2\alpha_3$, $H_3 = 1$.

For brevity, we will use the notation

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$$\dot{f}_k = \frac{\partial f_k}{\partial t}, \quad f_{k,l} = \frac{\partial f_k}{\partial \alpha_i}, \quad f_{k,ij} = \frac{\partial^2 f_k}{\partial \alpha_i \partial \alpha_j} \quad \text{and so on}$$

2. In the chosen system of coordinates, the force and couple-stresses σ_{ji} and μ_{ji} are given by [5–7]

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij} \quad (ML^{-1}T^{-2}) \tag{2.1}$$

$$\mu_{ji} = (\gamma+\varepsilon)\chi_{ji} + (\gamma-\varepsilon)\chi_{ij} + \beta\chi_{kk}\delta_{ij} \quad (MT^{-2})$$

The strain tensor and bending-torsion tensor have the representations (with summation over k)

$$\gamma_{ji} = \frac{1}{H_j}u_{i,j} - \frac{u_j}{H_j H_i}H_{j,i} + \delta_{ji}\frac{u_k}{H_j H_k}H_{j,k} - \epsilon_{jik}\,\omega_k$$

$$\chi_{ji} = \frac{1}{H_j}\omega_{i,j} - \frac{\omega_j}{H_j H_i}H_{j,i} + \delta_{ji}\frac{\omega_k}{H_j H_i}H_{j,k} \tag{2.2}$$

$\mu = E/[2(1+\nu)]$, $\lambda = \nu E/[1+\nu)(1-2\nu)$ are Lamé constants, E is the modulus of elasticity, ν is Poisson's ratio, α, γ, E, β are new constants of elasticity ($\mu, \lambda, E, \alpha - ML^{-1}T^{-2}$; $\gamma, \varepsilon, \beta - ML^{-2}$), u_i are the components of the displacement of any point of the shell, δ_{ji} is the Kronecker delta and ϵ_{jik} is the Levi–Civita tensor.

The equations of motion can be written as follows:

$$\begin{aligned}
&(H_2\sigma_{11})_{,1} + (H_1\sigma_{21})_{,2} + (H_1H_2\sigma_{31})_{,3} + H_{1,2}\sigma_{12} + H_2H_{1,3}\sigma_{13} - H_{2,1}\sigma_{22} + H_1H_2X_1 = \rho H_1H_2\ddot{u}_1 \\
&(H_1\sigma_{22})_{,2} + (H_1H_2\sigma_{32})_{,3} + (H_2\sigma_{12})_{,1} + H_{2,1}\sigma_{21} + \\
&+H_1H_{2,3}\sigma_{23} - H_{1,2}\sigma_{11} + H_1H_2X_2 = \rho H_1H_2\ddot{u}_2 \\
&(H_1H_2\sigma_{33})_{,3} + (H_2\sigma_{13})_{,1} + (H_1\sigma_{23})_{,2} - H_1H_{2,3}\sigma_{22} - H_2H_{1,3}\sigma_{11} + H_1H_2X_3 = rH_1H_2\ddot{u}_3
\end{aligned} \tag{2.3}$$

Also

$$\begin{aligned}
&(H_2\mu_{11})_{,1} + (H_1\mu_{21})_{,2} + (H_1H_2\mu_{31})_{,3} + H_{1,2}\mu_{12} + H_2H_{1,3}\mu_{13} - H_{2,1}\mu_{22} + \\
&+H_1H_2Y_1 + H_1H_2(\sigma_{23} - \sigma_{32}) = JH_1H_2\ddot{\omega}_1 \\
&(H_1\mu_{22})_{,2} + (H_1H_2\mu_{32})_{,3} + (H_2\mu_{12})_{,1} + H_{2,1}\mu_{21} + H_1H_{2,3}\mu_{23} - H_{1,2}\mu_{11} + \\
&+H_1H_2Y_2 + H_1H_2(\sigma_{31} - \sigma_{32}) = JH_1H_2\ddot{\omega}_2 \\
&(H_1H_2\mu_{33})_{,3} + (H_2\mu_{13})_{,1} + (H_1\mu_{23})_{,2} - H_1H_{2,3}\mu_{22} - H_2H_{1,3}\mu_{11} + \\
&+H_1H_2Y_3 + H_1H_2(\sigma_{12} - \sigma_{21}) = JH_1H_2\ddot{\omega}_3
\end{aligned} \tag{2.4}$$

Here X_i, Y_i are the components of mass forces and moments, respectively, r is the density (ML^{-3}) and J is a dynamic characteristic of the medium (a measure of the inertia during rotation (ML^{-1}).

These equations have been written ignoring the hypotheses and assumptions described above. They are quite general and do not appear to have been published in papers on the theory of shells. Obviously, they become much simpler in the case of very shallow shells.

3. With the accuracy of the theory of very hollow shells [1–3] and the given hypotheses and assumptions, from (2.2) by (1.1) and (1.3) we have

$$\gamma_{33} = u_{3,3} \approx 0, \quad u_3 = w(\alpha_1, \alpha_2) \tag{3.1}$$

$$\gamma_{3n} = u_{n,3} + w_{,n} - f(\alpha_3)\psi_n, \quad \gamma_{n3} = -k_n u_n + f(\alpha_3)\psi_n, \quad n = 1,2 \tag{3.2}$$

Substituting the values of the stresses σ_{3n} from (1.2) and the corresponding strains from (3.2) into (2.1), we solve the resulting equations for u_i, taking $u_1 = u(\alpha_1, \alpha_2)$, $u_2 = \nu(\alpha_1, \alpha_2)$ with $\alpha_3 = 0$, and obtain

$$u_n = u(v) - \alpha_3 w_{,n} + \frac{I_0}{\mu+\alpha}(\varphi_n + 2\alpha\psi_n), \quad n = 1,2 \tag{3.3}$$

$$I_0 = \int_0^{\alpha_3} f(\alpha_3)d\alpha_3 = \frac{\alpha_3}{2}\left(\frac{h^2}{4} - \frac{\alpha_3^2}{3}\right) \tag{3.4}$$

Thus, by virtue of the above hypotheses, the displacements of any point of the shell (3.1), (3.3) can be represented by means of seven unknown "two-dimensional" functions u, v, w, φ_1, φ_2, ψ_1, ψ_2 (with arguments α_1, α_2). In addition, there is one unknown "two-dimensional" function $\psi_3(\alpha_1, \alpha_2)$, which represents rotations ω_3.

The use of (1.1), (3.1) and (3.3) basically reduces a three-dimensional problem in the asymmetric theory of elasticity to a two-dimensional problem in the theory of shells.

Substituting the values γ_{ji} and χ_{ji} into (2.1) and using (1.1), (1.3), (3.1) and (3.3), as well as (1.2), we obtain the following equations for the stresses (any relations not written out are obtained by cyclical replacement of the symbols in brackets)

$$\left.\begin{aligned}\sigma_{11} &= B\{u_{,1} + k_1 w + \nu(\upsilon_{,2} + k_2 w) - \alpha_3(w_{,11} + \nu w_{,22}) + \\ &+ \frac{I_0 B}{\mu+\alpha}[\varphi_{1,1} + \nu\varphi_{2,2} + 2\alpha(\psi_{1,1} + \nu\psi_{2,2})]\end{aligned}\right\} \tag{3.5}$$

$$(1 \leftrightarrow 2, u \leftrightarrow \nu), \quad B = E/(1-\nu^2), \quad B_{12} = \nu E/(1-\nu^2)$$

$$\begin{aligned}\sigma_{12} &= (\mu+\alpha)\nu_{,1} + (\mu-\alpha)u_{,2} - \alpha_3 2\mu w_{,12} - 2\alpha\psi_3 + I_0 \times \\ &\times\left[\varphi_{2,1} + \frac{\mu-\alpha}{\mu+\alpha}\varphi_{1,2} + 2\alpha\left(\psi_{2,1} + \frac{\mu-\alpha}{\mu+\alpha}\psi_{1,2}\right)\right] \quad (1 \leftrightarrow 2, \ u \leftrightarrow \nu, \ \psi_3 \leftrightarrow -\psi_3)\end{aligned} \tag{3.6}$$

$$\begin{aligned}\sigma_{,3} &= -k_n(\mu+\alpha)u + k_n(\mu+\alpha)\alpha_3 w_{,n} + 2f(\alpha_{,3})\alpha\psi_n + \\ &+\left[f(\alpha_3)\frac{\mu-\alpha}{\mu+\alpha} - k_n I_0\right](\varphi_n + 2\alpha\psi_n) \quad (n=1,2; \ u \leftrightarrow \nu)\end{aligned} \tag{3.7}$$

We also have

$$\mu_{11} = 2\gamma w_{,12} - f(\alpha_3)[(2\gamma+\beta)\psi_{2,1} - \beta\psi_{1,2}] + [2\gamma k_1 + \beta(k_1 + k_2)]\psi_3 \tag{3.8}$$

$$(1 \leftrightarrow 2, \ w \leftrightarrow -w, \ f \leftrightarrow -f)$$

$$\mu_{12} = -(\gamma+\varepsilon)w_{,11} + (\gamma-\varepsilon)w_{,22} + f(\alpha_3)[(\gamma+\varepsilon)\psi_{1,1} - (\gamma-\varepsilon)\psi_{2,2}] \tag{3.9}$$

$$(1 \leftrightarrow 2, \ w \leftrightarrow -w, \ f \leftrightarrow -f)$$

$$\begin{aligned}\mu_{13} &= (\gamma+\varepsilon)(\psi_{3,1} - k_1 w_{,2}) + [k_1 f(\alpha_3)(\gamma+\varepsilon) + \alpha_3(\gamma-\varepsilon)]\psi_2 \\ \mu_{23} &= (\gamma+\varepsilon)(\omega_{3,2} + k_2 w_{,1}) - [k_2 f(\alpha_3)(\gamma+\varepsilon) + \alpha_3(\gamma-\varepsilon)]\psi_1\end{aligned} \tag{3.10}$$

The couple-stresses μ_{33}, μ_{3i} which, in thin shells, are quite small compared with μ_{ii}, μ_{ik}, can, if necessary, be determined from the equations of motion (2.4) allowing for the surface conditions $\mu_{33} = \mu_{31} = \mu_{32} = 0$ at $\alpha_3 = \pm h/2$.

The stresses (3.5)–(3.10) in the cross-sections of the shell produce internal forces and moments which, per unit length of the middle surface and to the usual accuracy of the theory of very hollow shells [1–3], can be written as follows:

tangential and shearing forces due to force stresses

$$\begin{aligned}T_{11} &= Bh[u_{,1} + k_1 w + \nu(\upsilon_{,2} + k_2 w)] \quad (1 \leftrightarrow 2, \ u \leftrightarrow \upsilon) \\ S_{12} &= h[(\mu+\alpha)\upsilon_{,1} + (\mu-\alpha)u_{,1} - 2\alpha\psi_3] \quad (1 \leftrightarrow 2, \ u \leftrightarrow \upsilon, \ \psi_3 \leftrightarrow -\psi_3) \\ N_{n3} &= -k_n(\mu+\alpha)hu + \frac{h^3}{12}\left(\frac{\mu-\alpha}{\mu+\alpha}\varphi_n + \frac{4\mu\alpha}{\mu+\alpha}\psi_n\right) \quad (n=1,2; \ u \leftrightarrow \upsilon)\end{aligned} \tag{3.11}$$

bending moments and torques due to force stresses

$$M_{11}=-\frac{Bh^3}{12}(w_{,11}+\nu w_{,22})+\frac{Bh^5}{120(\mu+\alpha)}[\varphi_{1,1}+\nu\varphi_{2,2}+2\alpha(\psi_{1,1}+\nu\psi_{2,2})] \quad (1\leftrightarrow 2)$$

$$H_{12}=-2\mu\frac{h^3}{12}w_{,12}+\frac{h^5}{120}\left[\varphi_{2,1}+\frac{\mu-\alpha}{\mu+\alpha}\varphi_{1,2}+2\alpha\left(\psi_{2,1}+\frac{\mu-\alpha}{\mu+\alpha}\psi_{1,2}\right)\right] \quad (1\leftrightarrow 2) \tag{3.12}$$

total torques and bending moments due to couple-stresses

$$P_{11}=2\gamma hw_{,12}-\frac{h^3}{12}[(2\gamma+\beta)\psi_{2,1}-\beta\psi_{1,2}] \quad (1\leftrightarrow 2,\ w\leftrightarrow -w,\ h\leftrightarrow -h)$$

$$R_{12}=h[(\gamma-\varepsilon)w_{,22}-(\gamma+\varepsilon)w_{,11}]+\frac{h^3}{12}[(\gamma+\varepsilon)\psi_{1,1}-(\gamma-\varepsilon)\psi_{2,2}] \quad (1\leftrightarrow 2,\ h\leftrightarrow -h) \tag{3.13}$$

$$Q_{13}=k_1(\gamma+\varepsilon)\left[\frac{h^3}{12}\psi_2-hw_{,2}\right]+(\gamma+\varepsilon)h\psi_{3,1} \quad (1\leftrightarrow 2,\ h\leftrightarrow -h,\ \psi_3\leftrightarrow -\psi_3)$$

4. Averaging the equations of motion (2.3), (2.4) over the shell thickness, using (1.1)–(1.3), (3.4)–(3.13) and the conditions on the surfaces $\alpha_3=\pm h/2$, with the usual accuracy of shallow shell theory [1–3] we obtain the following equations of motion in forces and moments

$$\begin{gathered}T_{11,1}+S_{21,2}+k_1N_{13}=\rho h\ddot{u} \quad (1\leftrightarrow 2,\ u\leftrightarrow v)\\ N_{13,1}+N_{23,2}-k_1T_{11}-k_2T_{22}=-(Z^++Z^-)+\rho h\ddot{w}\\ M_{11,1}+H_{21,2}-\frac{h^3}{12}\varphi_1-k_1^2\frac{h^5}{120}2\alpha\psi_1=-\rho\frac{h^3}{12}\ddot{w}_{,1}+\frac{\rho h^5}{120(\mu+\alpha)}(\ddot{\varphi}_1+2\alpha\ddot{\psi}_1) \quad (1\leftrightarrow 2)\end{gathered} \tag{4.1}$$

and also

$$\begin{gathered}P_{11,1}+R_{21,2}+k_1Q_{13}+N_{23}-\frac{h^3}{12}\varphi_2=Jh\ddot{w}_{,2}-J\frac{h^3}{12}\ddot{\psi}_2 \quad (1\leftrightarrow 2,\ h\leftrightarrow -h,\ N_{23}\leftrightarrow -N_{13})\\ Q_{13,1}+Q_{23,2}-k_1P_{11}-k_2P_{22}+S_{12}-S_{21}=Jh\ddot{\psi}_3\end{gathered} \tag{4.2}$$

Substituting the values of the internal forces and moments into (4.1) and (4.2), we obtain a complete system of eight differential equations in the eight unknown functions u, v, w, φ_1, φ_2, ψ_1, ψ_2, ψ_3

$$\begin{aligned}&Bu_{,11}+(\mu+\alpha)u_{,22}+(B_{12}+\mu-\alpha)v_{,12}+B(k_1+\nu k_2)w_{,1}+\\ &+k_1\frac{h^2}{12}\left(\frac{\mu-\alpha}{\mu+\alpha}\varphi_1+\frac{4\mu\alpha}{\mu+\alpha}\psi_1\right)+2\alpha\psi_{3,2}=\rho\ddot{u} \quad (1\leftrightarrow 2,\ u\leftrightarrow v,\ \psi_3\leftrightarrow -\psi_3)\end{aligned} \tag{4.3}$$

$$\begin{aligned}&[k_1(B+\mu+\alpha)+k_2B_{12}]u_{,1}+[k_1(B+\mu+\alpha)+k_1B_{12}]v_{,2}+\\ &+(k_1^2B+2k_1k_2B_{12}+k_2^2B)w-\frac{h^2}{12}\left[\frac{\mu-\alpha}{\mu+\alpha}(\varphi_{1,1}+\varphi_{2,2})+\right.\\ &\left.+\frac{4\mu\alpha}{\mu+\alpha}(\psi_{1,1}+\psi_{2,2})\right]=\frac{Z}{h}-\rho\ddot{w}, \quad Z=Z^++Z^-\end{aligned} \tag{4.4}$$

$$\begin{aligned}&Bw_{,111}+Bw_{,122}-\frac{h^2}{10}\left[\frac{B}{\mu+\alpha}\varphi_{1,11}+\varphi_{1,22}+\frac{B_{12}+\mu-\alpha}{\mu+\alpha}\varphi_{2,12}+\right.\\ &\left.+2\alpha\left(\frac{B}{\mu+\alpha}\psi_{1,11}+\psi_{1,22}+\frac{B_{12}+\mu-\alpha}{\mu+\alpha}\psi_{2,12}\right)\right]+\varphi_1+k_1^2\frac{h^2}{10}2\alpha\psi_1=\\ &=\rho\ddot{w}_{,1}-\frac{\rho h^2}{10(\mu+\alpha)}(\ddot{\varphi}_1+2\alpha\ddot{\psi}_1) \quad (1\leftrightarrow 2)\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&(\gamma+\varepsilon)(w_{,111}+w_{,122})-\frac{h^2}{12}[(\gamma+\varepsilon)\psi_{1,11}+(2\gamma+\beta)\omega_{1,22}-(\gamma-\varepsilon+\beta)\psi_{2,12}]-\\
&-[k_2(3\gamma+\varepsilon)+(k_1+k_2)\beta]\psi_{3,2}-\frac{h^2}{12}\frac{2\alpha}{\mu+\alpha}(\varphi_1-2\alpha\psi_1)-k_1(\mu+\alpha)u+\\
&+k_2^2\frac{h^2}{12}(\gamma+\varepsilon)\psi_1=J\ddot{w}_{,1}-J\frac{h^3}{12}\ddot{\psi}_1 \quad (1\leftrightarrow 2, u\leftrightarrow v, \psi_3\leftrightarrow -\psi_3)
\end{aligned}\tag{4.6}
$$

$$
\begin{aligned}
&2\alpha(v_{,1}-u_{,2})-(k_1-k_2)(3\gamma+\varepsilon)w_{,12}+(\gamma+\varepsilon)(\psi_{3,11}+\psi_{3,22})+\\
&+\frac{h^2}{12}[k_1(3\gamma+\beta+\varepsilon)+k_2\beta]\psi_{2,1}-\frac{h^2}{12}[k_2(3\gamma+\beta+\varepsilon)+k_1\beta]\psi_{1,2}-\\
&-[(2\gamma+\beta)(k_1^2+k_2^2)+2\beta k_1k_2+4\alpha]\psi_3=J\ddot{\psi}_3
\end{aligned}\tag{4.7}
$$

To these must be added the boundary conditions at the ends of the shell. The averaged boundary conditions can be written like the boundary conditions of the improved theory [1, 2].

5. For a plate ($k_1 = 0, k_2 = 0$), the system of resolvents splits into two independent systems. The first consists of the three equations (4.3) and (4.7) in the three unknowns u, v, ψ_3 (the plane problem). The second consists of the five equations (4.4), (4.5) and (4.6) in the five unknowns w, φ_1, φ_2, ψ_1, ψ_2 (transverse bending). In the special case when $\alpha = 0$, the first five equations are the same as in the improved theory [1, 2], while the other three can only be associated with the former in terms of the unknown displacements u, υ and w.

6. We will consider the model problem of a circular cylindrical shell with radius of curvature $R_2 = R$ which is spherically supported at its ends ($\alpha_1 = 0, \alpha_2 = a$) and is axisymmetrically loaded ($Z = q\sin\lambda\alpha_1$, $\lambda = \pi/\alpha$). In this case, obviously, all the unknown quantities will be functions of $v = 0$, $\varphi_2 = 0$, $\psi_2 = 0$, $\psi_3 = 0$ only. Also $v = 0$, $\varphi_2 = 0$, $\psi_2 = 0$, $\psi_3 = 0$.

The system of resolvents in the required functions $u(\alpha_1)$, $w(\alpha_1)$, $\varphi_1 = \varphi(\alpha_1)$, $\psi_1 = \psi(\alpha_1)$ takes the form (the prime denotes a derivative with respect to α_1)

$$
\begin{aligned}
&Bu''+B_{12}\frac{1}{R}w'=0\\
&B_{12}\frac{1}{R}u'+B\frac{w}{R^2}-\frac{h^2}{12}\left(\frac{\mu-\alpha}{\mu+\alpha}\varphi'+\frac{4\mu\alpha}{\mu+\alpha}\psi'\right)=\frac{q}{n}\sin\lambda\alpha_1\\
&Bw''-\frac{h^2}{10}\frac{B}{\mu+\alpha}(\varphi''+2\alpha\psi'')+\varphi=0\\
&(\gamma+\varepsilon)w'''-\frac{h^2}{12}\left[(\gamma+\varepsilon)\psi''-\frac{4\mu\alpha}{\mu+\alpha}\psi+\frac{2\alpha}{\mu+\alpha}\varphi-\frac{\gamma+\varepsilon}{R^2}\psi\right]=0
\end{aligned}\tag{6.1}
$$

Table 1

b^2	$\frac{\alpha}{\mu}$	$\frac{B}{hq}w_0$	$\frac{B}{hq}u_0$	$\frac{h^2}{q}\varphi_0$	$\frac{Bh^2}{q}\psi_0$
	0.0	430	41.1	1.65	
4	0.05	398	38.0	1.51	87.0
	0.1	396	37.9	1.50	49.2
	0.5	395	37.7	1.50	15.6
	0.05	398	38.0	1.51	145
2	0.1	396	37.9	1.50	84.4
	0.5	395	37.7	1.50	26.4
	0.05	349	33.3	1.29	218
1	0.1	335	32.1	1.23	134
	0.5	322	30.7	1.17	43.0

For internal forces and moments we have

$$
\begin{aligned}
&T_{11}=Bh\left(u'+\nu\frac{w}{R}\right),\quad T_{22}=Bh\left(\frac{w}{R}+\nu u'\right)\\
&N_{13}=\frac{h^3}{12}\left(\frac{\mu-\alpha}{\mu+\alpha}\varphi+\frac{4\mu\alpha}{\mu+\alpha}\psi\right),\quad M_{22}=\nu M_{11}\\
&M_1=-\frac{Bh^3}{12}w''+\frac{Bh^5}{120(\mu+\alpha)}(\varphi'+2\alpha\psi),\quad R_{21}=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}R_{12}\\
&R_{12}=(\gamma+\varepsilon)\left(\frac{h^3}{12}\psi_{,1}-hw''\right),\quad Q_{23}=(\gamma+\varepsilon)\left(\frac{h}{2}w'-\frac{h^3}{12R}\psi\right)
\end{aligned}
\tag{6.2}
$$

Assuming

$$
u=u_0\cos\lambda\alpha_1,\quad w=w_0\sin\lambda\alpha_1,\quad \varphi=\varphi_0\cos\lambda\alpha_1,\quad \psi=\psi_0\cos\lambda\alpha_1
$$

the conditions of free support at the ends of the shell are satisfied, and from Eqs (6.1) we obtain a system of algebraic equations from which the required coefficients u_0, w_0, φ_0, ψ_0 can be found.

The analytic representations of these quantities are long and cumbersome, so we will only give the results for some specific numerical cases. Let $a = R$, $h/a = h/R = 1/20$, $\nu = 0.3$, $\mu = 0.3b$. Further, we assume [8–10] that $\gamma + \varepsilon = 4\mu l^2$, where l is a new constant of the material with the dimension of length. It is obvious in this case that $\gamma = 2\mu l^2(1 + \eta)$, $\varepsilon = 2\mu l^2(1 - \eta)$. The dimensionless constant η, as we know [10], varies between -1 and 1 and is assumed [8, 10] to be not very different from the value of $-\nu$. In these examples the new constant of elasticity a takes the values: zero, 0.05μ, 0.1μ and 0.5μ, and the dimensionless quantity $b = h/l$, which is the ratio of the shell thickness to a parameter of the material, takes the values: 1, 1.14142 and 2. The results are given in the table.

The required values can be found with the improved theory [1, 2] when $b^2 = 4$, $\alpha/\mu = 0.0$, but normally one is limited to quite thin shells $h/\alpha = 1/20$, for which the influence of transverse shears on the values is reduced considerably, thus making the effects due to asymmetric elasticity more obvious.

The table shows that even quite small values of the new constants of elasticity can result in appreciably different values from those obtained in the classical theory. Moreover, for a fixed value of l, a change in a produces a change of only one-tenth of that amount in the values of u_0, w_0, φ_0, which is important in the design of experiments to determine the new (basic) constant of elasticity l. By reducing the shell thickness, we increase the influence on the results of the new constant of elasticity l. In particular, for fixed l, when the absolute thickness of the shell is reduced, there is considerable disagreement between the results of the classical (improved) theory and the theory proposed here. Assuming, for example, that $b = 1$, with a shell thickness $h = 0.05$, which is equivalent to taking $l = 0.05$, we conclude that the required values of u_0, w_0, φ_0 will differ from the classical values by 20–25%.

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